MODEL-CHECKS FOR HOMOSCHEDASTIC SPATIAL LINEAR REGRESSION MODEL BASED ON BOOTSTRAP METHOD

W. Somayasa

Mathematics Department of Haluoleo University, Kendari

email: wayan.somayasa@yahoo.de

Abstract

In this paper we propose Efron residual based-bootstrap approximation methods in asymptotic model-checks for homoschedastic spatial linear regression models. It is shown that under some regularity conditions given to the known regression functions the bootstrap version of the sequence of least squares residual partial sums processes converges in distribution to a centred Gaussian process having sample paths in the space of continuous functions on $I := [0,1] \times [0,1]$. Thus, Efron residual based-bootstrap is a consistent approximation in the usual sense. The finite sample performance of the bootstrap level α Kolmogorov-Smirnov (KS) type test is also investigated by means of Monte Carlo simulation.

Key words: residual based-bootstrap, asymptotic model-check, homoschedastic spatial linear regression models, partial sums, Gaussian process.

I. Introduction

Practically the correctness of the assumed linear models is usually evaluated by analysing the cumulative sums (CUSUM) of the least squares residuals. To this end a huge amount of literature is available, see among others MacNeill (1978) or Bischoff and Miller (2000) for one dimensional context. In the spatial context we refer the reader to MacNeill and Jandhyala (1993) and Xie and MacNeill (2006).

To see the above mentioned problem in more detail let us consider an experiment performed on an experimental region $I := [0,1] \times [0,1]$ under n^2 experimental conditions taken from a regular lattice, given by

$$\Xi_n := \{(\ell/n, k/n) : 1 \le \ell, k \le n\} \subset I,\ n \ge 1. \tag{1}$$

Accordingly, we put together the observations carried out in Ξ_n in an $n \times n$ matrix $Y_{n\times n} := (Y_{\ell k})_{k=1,\ell=1}^{n,n} \in \Re^{n\times n}$, where for $1 \le \ell, k \le n$, $Y_{\ell k}$ is the observation at the point $(\ell/n, k/n)$, and $\Re^{n\times n}$ is the space of $n \times n$ real matrices furnished with the Euclidean inner product $\langle A, B\rangle_{\Re^{n\times n}} := \mathrm{trace}(A^t B)$ and the corresponding norm given by $\|A\|_{\Re^{n\times n}} = \sqrt{\mathrm{trace}(A^t A)}$, for $A, B \in \Re^{n\times n}$. It is worth noting that for our model we take I as the experimental region without loss of generality instead of any compact subset of $\Re^2$. For any $f : I \to \Re$, let $f(\Xi_n) := (f(\ell/n, k/n))_{k=1,\ell=1}^{n,n} \in \Re^{n\times n}$, then we have $Y_{n\times n} = g(\Xi_n) + E_{n\times n}$, where g is a true but unknown real valued regression function having bounded variation on I and

$E_{n\times n} := (\varepsilon_{\ell k})_{k=1,\ell=1}^{n,n}$ is the corresponding matrix of independent and identically distributed random errors having mean zero and variance $\sigma^2 \in (0,\infty)$, for $1 \le \ell, k \le n$. We are interested in testing the hypotheses

$$H_0 : g(\Xi_n) \in W_n \quad \text{vs.} \quad K : g(\Xi_n) \notin W_n,\; n \ge 1, \tag{2}$$

where $W_n := [f_1(\Xi_n),\ldots,f_p(\Xi_n)] \subset \Re^{n\times n}$ is a sub space of $\Re^{n\times n}$ generated by $f_1(\Xi_n)$, … , $f_p(\Xi_n)$.

Under H_0, the matrix of least squares residuals is given by $R_{n\times n} := pr_{W_n^\perp} Y_{n\times n} = pr_{W_n^\perp} E_{n\times n}$ (Stapleton, 1995), where $pr_{W_n^\perp}$ is the orthogonal projector onto the orthogonal complement of W_n. Without loss of the generality we assume throughout this work that $f_1(\Xi_n)$, … , $f_p(\Xi_n)$ is an orthogonal basis of W_n. Hence, by the elementary linear algebra we get under H_0,

$$R_{n\times n} := E_{n\times n} - \sum_{i=1}^{p} \frac{\langle f_i(\Xi_n), E_{n\times n}\rangle_{\Re^{n\times n}} f_i(\Xi_n)}{\langle f_i(\Xi_n), f_i(\Xi_n)\rangle_{\Re^{n\times n}}}. \tag{3}$$

Equivalently by the vec operator, we have

$$\operatorname{vec}(R_{n\times n}) = \operatorname{vec}(E_{n\times n}) - \left(X_n (X_n^t X_n)^{-1} X_n^t\right) \operatorname{vec}(E_{n\times n}),$$

where X_n is the design matrix of the model under H_0, i.e., an $n^2 \times p$ matrix whose k^{th} column is given by the n^2 dimensional vector $\operatorname{vec}(f_k(\Xi_n))$, $k = 1,\ldots,p$. A consistent estimator for σ^2 is given by $\hat\sigma_n^2 := \frac{\|R_{n\times n}\|^2_{\Re^{n\times n}}}{n^2 - p}$ in the sense $\hat\sigma_n^2$ converges in probability to σ^2, as $n \to \infty$.

Suppose that the regression functions f_1, … , f_p are linearly independent and squared integrable on I with respect to the Lebesgue measure λ_I, then under H_0, MacNeill and Jandhyala (1993) and Xie and MacNeill (2006) showed that

$$\frac{1}{n\sigma} T_n(\operatorname{vec}(R_{n\times n}))(\cdot) \xrightarrow{D} B_f(\cdot) \text{ in } C(I), \text{ as } n \to \infty,$$

where for $(t,s) \in I$,

$$B_f(t,s) := B_2(t,s) - \left(\int_{[0,t]\times[0,s]} f^t d\lambda_I\right) G^{-1} \left(\overset{(R)}{\int_I} f dB_2\right), \tag{4}$$

and $T_n : \Re^{n^2} \to C(I)$ is the partial sums operator defined in Park (1971). Here $G := \left(\int_I f_k f_\ell \, d\lambda_I\right)_{k=1,\ell=1}^{p,p} \in \Re^{p\times p}$, B_2 is the standard Brownian sheet having sample pats in $C(I)$ and $\overset{(R)}{\int}$ stands for the Riemann-Stieltjes integral.

If t_n is a realisation of the Kolmogorov-Smirnov (KS) statistics $KS_{n,f} := \max_{0\le\ell,k\le n}\left|\frac{1}{n\sigma}T_n(\text{vec}(R_{n\times n}))(\ell/n,k/n)\right|$, then a level α test of KS type based on residual partial sums processes for testing (2) will reject H_0 if and only if $t_n \ge c_{1-\alpha}$, where $c_{1-\alpha}$ is a constant that satisfies the equation

$$P_{H_0}\left\{\max_{0\le\ell,k\le n}\left|\frac{1}{n\sigma}T_n(\text{vec}(R_{n\times n}))(\ell/n,k/n)\right|\ge c_{1-\alpha}\right\}=\alpha .$$

Hence, by the continuous mapping theorem (Billingsley, 1968, p. 29-30), $c_{1-\alpha}$ can be approximated by means of the distribution of $\sup_{0\le t,s\le 1}|B_f(t,s)|$.

As it is presented in (4), the limiting distribution of the KS type statistics under H_0 is not mathematically tractable, because it depends on the structure of the designs and the regression functions. Therefore the application of the preceding tests procedure is limited. It is the purpose of the preceding paper to investigate the performance of bootstrap test of (2). Although the application of the bootstrap appears naturally in the present context, to the knowledge of the author this topic has not found much attention in the literature. One purpose of the present paper is to demonstrate that Efron residual based-bootstrap (Shao and Tu, 1995) provides simple and reliable alternative for constructing asymptotic critical region of such a KS type test. This will be discussed in the Section 2. In Section 3 we develop Monte Carlo simulation in investigating the finite sample characteristics of the bootstrap test. An application of the developed method to a real data will be presented in Section 4. Finally we close the paper in Section 5 presenting conclusion and some remarks for future researches.

II. Bootstrap Approximation

Let $r_{\ell k}$ be the component of the residual matrix in the k^{th} row and ℓ^{th} column computed based on the original observations $Y_{n\times n}$ under $H_0, 1\le \ell,k\le n$. We define an $n\times n$ matrix of centred residuals $\hat{R}_{n\times n} := (r_{\ell k}-\bar{r})_{k=1,\ell=1}^{n,n}$, where $\bar{r} := \frac{1}{n^2}\sum_{k=1}^{n}\sum_{\ell=1}^{n} r_{\ell k}$. Further, let $F_{\hat{R}}$ be the empirical distribution function of the components of $\hat{R}_{n\times n}$ putting mass $\frac{1}{n^2}$ to $r_{\ell k}-\bar{r}$, for $1\le \ell,k\le n$, i.e., for $x\in\Re$, $F_{\hat{R}}(x) := \frac{1}{n^2}\sum_{k=1}^{n}\sum_{\ell=1}^{n} 1_{(-\infty,x]}(r_{\ell k}-\bar{r})$, where 1_A is the indicator function of A. Efron residual based-bootstrap defines the bootstrap observations by

$$Y^*_{n\times n} := (Y^*_{\ell k})_{k=1,\ell=1}^{n,n} = \overline{g(\Xi_n)} + E^*_{n\times n},$$

where $E^*_{n\times n} := (\varepsilon^*_{\ell k})_{k=1,\ell=1}^{n,n}$ is a random matrix whose components are a random sample from $F_{\hat{R}}$ and

$$\overline{g(\Xi_n)} = \sum_{i=1}^{p}\frac{\langle f_i(\Xi_n), Y_{n\times n}\rangle_{\Re^{n\times n}}\, f_i(\Xi_n)}{\langle f_i(\Xi_n), f_i(\Xi_n)\rangle_{\Re^{n\times n}}} \tag{5}$$

is the ordinary least squares estimator of $g(\Xi_n)$ under H_0. Let E^* and var^* be the expectation and variance operators conditioned on $F_{\hat{R}}$. Then by the construction it is clear that $E^*(\varepsilon^*_{\ell k}) = 0$ and $\hat{\sigma}_n^{*2} := \mathrm{var}^*(\varepsilon^*_{\ell k}) = \frac{n^2-p}{n^2}\hat{\sigma}_n^2 - (\bar{r})^2$ which clearly converges in probability to σ^2 by the famous weak law of large number and Slutsky's theorem.

The bootstrap analogous of the residual partial sums process is given by

$$\frac{1}{n\hat{\sigma}_n^*} T_n(\mathrm{vec}(R^*_{n\times n}))(\cdot) = \frac{1}{n\hat{\sigma}_n^*} T_n(\mathrm{vec}(E^*_{n\times n}) - X_n(X_n^t X_n)^{-1} X_n^t \mathrm{vec}(E^*_{n\times n}))(\cdot). \quad (6)$$

Hence, the bootstrap version of the KS type statistics can be represented by

$$KS^*_{n,f} := \max_{0\le \ell,k\le n}\left|\frac{1}{n\hat{\sigma}_n^*} T_n(\mathrm{vec}(R^*_{n\times n}))(\ell/n, k/n)\right|. \quad (7)$$

Theorem 1

Suppose that the regression functions $f_1, \ldots, f_p$ are linearly independent in the space of squared integrable functions with respect to λ_I, denoted by $L_2(\lambda_I)$, continuous and have bounded variation on I. Then under H_0 it holds,

$$\frac{1}{n\hat{\sigma}_n^*} T_n(\mathrm{vec}(R^*_{n\times n}))(\cdot) \xrightarrow{D} B_f(\cdot) \text{ in } C(I), \text{ as } n\to\infty.$$

Proof

For any rectangle $A := [t_1, t_2] \times [s_1, s_2] \subseteq I$, $t_1 < t_2$, $s_1 < s_2$ and any real-valued function f on I, the increment of f over A is defined by

$$\Delta_I f := f(t_2, s_2) - f(t_2, s_1) - f(t_1, s_2) + f(t_1, s_1).$$

Further, for a fixed natural number n, let us define operators $U_n : C(I) \to \Re^{n^2}$ and $O_n : C(I) \to C(I)$, such that for any $u \in C(I)$ and $(t,s)\in I$,

$$U_n(u) := \mathrm{vec}\left(\Delta_{I_{\ell k}} u\right)_{k=1,\ell=1}^{n,n}, \quad (8)$$

$$O_n(u)(t,s) := T_n\left(U_n(u) - X_n(X_n^t X_n)^{-1} X_n^t U_n(u)\right)(t,s), \quad (9)$$

where $I_{\ell k} := [(\ell-1)/n, \ell/n] \times [(k-1)/n, k/n]$, $1 \le \ell, k \le n$. It is obvious that O_n is linear by the linearity of T_n. Moreover, since for every $a \in \Re^{n^2}$, $U_n(T_n(a)) = a$, it follows that O_n is idempotent. It can also be shown by the expression

$$O_n(u)(t,s) = T_n(U_n(u))(t,s) - \sum_{i=1}^{p}\frac{\langle \mathrm{vec}(f_i(\Xi_n)), U_n(u)\rangle T_n(\mathrm{vec}(f_i(\Xi_n)))(t,s)}{\langle \mathrm{vec}(f_i(\Xi_n)), \mathrm{vec}(f_i(\Xi_n))\rangle},$$

by the definition of norm of a operator on Banach space (Conway, 1985, p. 70), that

$$\|O_n\| \le 13\left(1 + 4\sum_{i=1}^{p}\|f_i\|_\infty^2 \Big/ \left(\inf_{0\le t,s\le 1} f_i(t,s)\right)^2\right) < \infty.$$

Thus, O_n is uniformly bounded on $C(I)$. This will directly imply to the continuity of O_n on $C(I)$, where $\|.\|_\infty$ is the supremum norm. Let us define an operator $O : C(I) \to C(I)$, such that for every $u \in C(I)$ and $(t,s) \in I$,

$$O(u)(t,s) := \Delta_{[0,t]\times[0,s]} u - \left(\int_{[0,t]\times[0,s]} f^t d\lambda_I \right) G^{-1} \left(\int_I^{(R)} f du \right).$$

The right hand side of (9) also can be written by

$$\begin{aligned} O_n(u)(t,s) &:= T_n\left(U_n(u)\right)(t,s) \\ &\quad - \left(\frac{1}{n^2} \gamma^t_{[nt][ns]} X_n \right) \left(\frac{1}{n^2} X_n^t X_n \right) \left(X_n^t U_n(u) \right), \end{aligned}$$

where $\gamma_{[nt][ns]} \in \Re^{n^2}$ is the vector whose first $[nt][ns]$ components are one and the reminder is zero. By the definition and by the continuity of u, we have

$$T_n\left(U_n(u)\right)(t,s) \xrightarrow{n\to\infty} \Delta_{[0,t]\times[0,s]} u, \ \forall (t,s) \in I. \tag{10}$$

Furthermore, since for $i = 1,\ldots,p$, f_i is bounded on I, then it holds component wise

$$\frac{1}{n^2} \gamma^t_{[nt][ns]} X_n \xrightarrow{n\to\infty} \left(\int_{[0,t]\times[0,s]} f_1 d\lambda_I, \ldots, \int_{[0,t]\times[0,s]} f_p d\lambda_I \right). \tag{11}$$

The convergence component wise

$$\left(\frac{1}{n^2} X_n^t X_n \right)^{-1} \xrightarrow{n\to\infty} \left(\left(\int_I f_i f_j d\lambda_I \right)_{i=1,\,j=1}^{p,p} \right)^{-1} = G^{-1}, \tag{12}$$

follows from the fact that the mapping $B \to B^{-1}$ is continuous on the space of invertible matrices. Further, since for $i = 1,\ldots,p$, f_i has bounded variation and u is continuous on I, the Riemann-Stieltjes integral of f_i with respect to u exists for all i. Consequently, we have the following convergence component wise

$$X_n^t U_n(u) \xrightarrow{n\to\infty} \left(\int_I^{(R)} f_1 du, \ldots, \int_I^{(R)} f_p du \right). \tag{13}$$

Thus, from (10), (11), (12) and (13), for every $u \in C(I)$, it holds $\|O_n(u) - O(u)\|_\infty \xrightarrow{n\to\infty} 0$. Next let $(u_n)_{n\ge1}$ be any sequence in $C(I)$ such that $\|u_n - u\|_\infty \xrightarrow{n\to\infty} 0$, then since O_n is bounded on $C(I)$, we finally have

$$\begin{aligned} 0 &\le \|O_n(u_n) - O(u)\|_\infty \\ &\le \|O_n\|_\infty \|u_n - u\|_\infty + \|O_n(u) - O(u)\|_\infty \xrightarrow{n\to\infty} 0. \end{aligned}$$

The last result shows that the only subset of $C(I)$ that satisfies the condition there exists a sequence $(u_n)_{n\ge1}$ that converges to u in $C(I)$, but the sequence $(O_n(u_n))_{n\ge1}$ does not converge to $O(u)$ is an empty set. Hence, by Theorem 5.5 in Billingsley (1968) and Theorem 4 in Park (1971), we get

$$\frac{1}{n\hat{\sigma}_n^*} T_n\left(vec(R^*_{n\times n})\right) = O_n\left(\frac{1}{n\hat{\sigma}_n^*} T_n\left(vec(E^*_{n\times n})\right)\right)$$

$$\xrightarrow{D} O(B_2), \text{ as } n \to \infty .$$

The proof of the theorem is complete since $B_2(t,s) = 0$, almost surely if $t = 0$ or $s = 0$.

Corollary 1

By the continuous mapping theorem, under H_0 we have $KS^*_{n,f} \xrightarrow{D} \sup_{0\le t,s\le 1} \left|B_f(t,s)\right|$, as $n \to \infty$.

Hence, for $\alpha \in (0,1)$, the constant $c_{1-\alpha}$ defined in page 4 can now be approximated by the constant $c^*_{1-\alpha}$ that satisfies the equation $P^*\left\{KS^*_{n,f} \ge c^*_{1-\alpha}\right\} = \alpha$. If t_n is a realisation of $KS_{n,f}$, then an asymptotic level α KS type test will reject H_0 if $t_n \ge c^*_{1-\alpha}$. We will illustrate in Section 4 the finite sample properties of this approach by means of simulation study.

Remark 3.

The extension of Theorem 1 to the case of $p \ge 3$ dimensional unit cube and experimental design $n_1 \times n_2 \times \cdots \times n_p$ regular lattice with $n_i \ne n_j$, for $i \ne j$ is straightforward.

III. Simulation

We develop Monte Carlo simulation for constructing the finite sample sizes critical region of the bootstrap level α test of the hypotheses

$$H_0 : g \in W \text{ vs. } K : g \in W' , \qquad (15)$$

where $W := [f_1, f_2, f_3]$ and $W' := [f_1, f_2, f_3, f_4, f_5, f_6]$, f_1, f_2, f_3, f_4, f_5, $f_6 : I \to \Re$, defined by $f_1(t,s) := 1$, $f_2(t,s) := t$, $f_3(t,s) := s$, $f_4(t,s) := t^2$, $f_5(t,s) := ts$, and $f_6(t,s) := s^2$. The simulation is developed according to the following algorithm.

Begin Algorithm

Step 1: Generate $n \times n$ matrix of pseudo random errors $E_{n\times n}$ from a distribution having mean zero and variance σ^2 .

Step 2: Compute the least squares residuals by the equation

$$vec(R_{n\times n}) = vec(E_{n\times n}) - X_n (X_n^t X_n)^{-1} X_n^t vec(E_{n\times n}) .$$

Step 3: Generate $n \times n$ matrix of bootstrap errors $E^*_{n\times n}$ by sampling with replacement from the centred residuals.

Step 4: Compute the bootstrap residuals by the equation

$$vec(R^*_{n\times n}) = vec(E^*_{n\times n}) - X_n (X_n^t X_n)^{-1} X_n^t vec(E^*_{n\times n}) .$$

Step 5: Compute $\hat{\sigma}^*_n$ and $KS^*_{n,f}$.

Step 6: Repeat Step 1 to Step 5 M times.

Step 7: Compute $c^*_{1-\alpha}$: sort M values of $KS^*_{n,f}$ obtained in Step 5 in ascending order, i.e.,

$$KS^{*(1)}_{n,f} \le KS^{*(2)}_{n,f} \le \cdots \le KS^{*(M)}_{n,f} ,$$

$$\text{then } c^*_{1-\alpha} = \begin{cases} KS_{n,f}^{*(M\times(1-\alpha))}\,;\ \text{if } M\times(1-\alpha)\in N \\ \\ KS_{n,f}^{*(M\times(1-\alpha)+1)}\,;\ \text{if } M\times(1-\alpha)\notin N \end{cases},$$

where N is the set of natural numbers.

End Algorithm

n	M=500	α					
		0.010	0.025	0.050	0.100	0.150	0.200
30	df_0	0.965	0.893	0.844	0.776	0.746	0.717
	df_1	1.183	1.119	1.057	0.994	0.954	0.928
	df_2	1.381	1.287	1.168	1.094	1.047	1.013
	df_3	1.327	1.240	1.150	1.092	1.026	0.970
40	df_0	0.934	0.868	0.816	0.757	0.730	0.697
	df_1	1.490	1.331	1.214	1.129	1.056	1.017
	df_2	1.304	1.232	1.184	1.093	1.030	0.989
	df_3	1.133	1.085	1.052	0.995	0.945	0.919
50	df_0	0.918	0.893	0.853	0.776	0.736	0.717
	df_1	1.316	1.234	1.165	1.081	1.023	0.993
	df_2	1.291	1.175	1.147	1.086	1.045	1.000
	df_3	1.336	1.220	1.173	1.098	1.047	1.014

Table 1. ***Simulated rejection probabilities of the Kolmogorov-Smirnov bootstrap test.***

For the simulation, we generate the error variables from i.i.d. random variables having mean zero and variance 0.5. Four cases are considered, that is for $1\le \ell, k\le n$,

$\varepsilon_{\ell k} \sim N(0,\ 0.5)$ and $\varepsilon_{\ell k} \sim \left(\chi^2_\nu - \nu\right)/\sqrt{4\nu}$, $\nu = 1,\ 2,\ 3$.

The simulation results are depicted in Table 1 for the sample sizes $n = 30, 40, 50$ and 60 and level $\alpha =$ 1%, 2.5%, 5%, 10% , 15%, and 20% with 500 replications. The entries in the row named df_0 are simulation results for which the errors are generated from $N(0,\ 0.5)$, whereas those in the rows named df_υ are generated from chi-square distribution having $\nu = 1,\ 2,\ 3$ degrees of freedom such that $E(\varepsilon_{\ell k}) = 0$ and $\text{var}(\varepsilon_{\ell k}) = 0.5$, for $1\le \ell, k\le n$. We note that for the simulation the generation of the random errors are restricted neither to a specific family of distributions nor to the distribution having variance 0.5 only. The variance as well as the family of the distribution may vary as long as they satisfy the conditions specified in Theorem 1.

IV. Example

As an example we consider the wheat-yield data (Mercer and Hall's data) presented in Cressie (1993), p. 454-455. The data are yield of grain (in pounds) observed over a 25 x 20 lattice of plots having 20 rows running from east to west and 25 columns of plots running from north to south.

The experiment consists of giving the 500 plots the same treatment (presumably fertilizer, water, etc.), from which it can be identified that the data are a realization of 500 independent random variables. As was informed in Cressie (1993), the plots are assumed to be equally spaced with the dimension of each plot being 10.82 ft by 8.05 ft. Figure 2 presents the perspective plot of the data.

Observing Figure 2, we postulate under H_0 a first order polynomial model as in Hypothesis (16). Since the variance is unknown, we use a consistent estimator $\hat{\sigma}^2_{nm}$. Calculated under H_0, the data give $\hat{\sigma}^2_{nm}$ = 0.1898, where

$$\hat{\sigma}^2_{nm} = \frac{\left\| Y_{n\times m} - \sum_{i=1}^{3} \frac{\langle f_i(\Xi_{nm}), Y_{n\times m}\rangle_{\Re^{n\times m}} f_i(\Xi_{nm})}{\langle f_i(\Xi_{nm}), f_i(\Xi_{nm})\rangle_{\Re^{n\times m}}} \right\|^2_{\Re^{n\times m}}}{nm-3},$$

$\Xi_{nm} = \{(\ell/25, k/20) : 1 \le \ell \le 25, 1 \le k \le 20\}$. The functions f_1, f_2, and f_3 are defined as in Section 4. After computation, we get the critical value of the KS type test 1.5919, so bootstrap approximation of the p-value of the test computed by simulation is given by $\hat{\alpha}$ = 0.0001. Thus it can be concluded that H_0 is rejected for all suitable values of α.

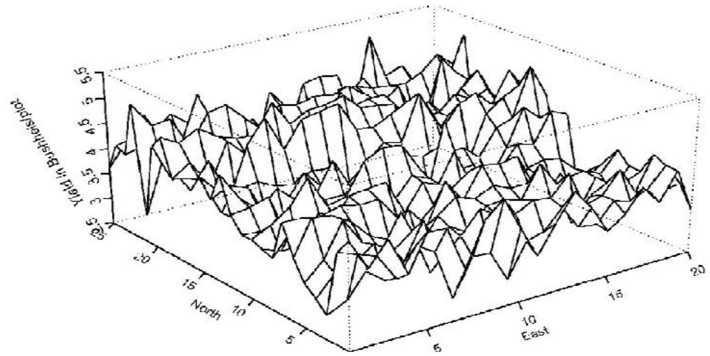

Figure 2. *The perspective plot of Mercer and Hall's data*

V. Concluding Remark

Efron residual based-bootstrap approximation to the KS type statistics based on least squares residual partial sums processes of homoschedastic spatial linear regression model is consistent. In the forthcoming paper we shall investigate the application of Efron as well as wild bootstrap in the case of heteroschedastic spatial linear regression model.

The experimental design considered so far is restricted to the regular lattice on the unit square I, since the prerequisite condition of the theory is satisfied well. In practice this situation is sometimes not reasonable, therefore we need to develop the theory in more general setting.

VI. References

1. Billingsley, P., 1968, *Convergence of Probability Measures*, John Wiley & Sons, Inc, New York, Singapore, Toronto.

2. Bischoff, W. and Miller, F., 2000, *Asymptotically Optimal Tests and Optimal Designs for Testing the Mean in Regression Models with Application to Change-Point Problems*, Ann. Inst. Statist. Math. Vol. 52, 658-679.
3. Conway, J.B., 1985, *A Course in Functional Analysis*, Springer-Verlag New York Inc., New York.
4. Cressie, Noel A. C., 1993, *Statistics for Spatial Data*, Revised Edition, John Wiley & Sons, Inc, New York, Singapore, Toronto.
5. MacNeill, I.B., 1978, *Properties of Sequences of Partial Sums of Polynomial Regression Residuals with Applications to Tests for Change of Regression at Unknown Times*, Ann. Statist. Vol. 6, 422-433.
6. MacNeill, I.B. and Jandhyala, V.K., 1993, *Change Point Methods for Spatial Data, Multivariate Environmental Statistics*, eds. by G.P. Patil and C.R. Rao, Elsevier Science Publishers B.V., 298-306.
7. Park, W.J., 1971, *Weak Convergence of Probability Measures on the Function Space* $C([0,1]^2)$, J. of Multivariate Analysis. Vol. 1, 433-444.
8. Shao, J. and Tu, D., 1995, *The Jackknife and Bootstrap*, Springer-Verlag, New York.
9. Stapleton, J. H., 1995, *Linear Statistical Model*, John Wiley & Sons Inc, New York, Toronto, Singapore.
10. Xie, L. and MacNeill, I.B., 2006, *Spatial Residual Processes and Boundary Detection*, South African *Statist. J*. Vol. **40**, no. **1**, 33-53.