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MODIFIKASI TEOREMA KONVOLUSI TRANSFORMASI KANONIKAL LINIER QUATERNION SISI KIRI

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ABSTRACT

Quaternion Linear Canonical Transform (QLCT) is a generalization of the Linear Canonical Transform (LCT) on quaternion algebra which plays an important role in optics and signal processing. QLCT can be seen as generalization of Quaternion Fourier Transform (QFT). Based on the fact, this paper propose the formulat of modification convolution theorem based on left-sided QLCT by considering the fundamental relationship between left-sided QLCT and QFT. The results showed the modification convolution theorem for left-sided QLCT based on relationship between left-sided QLCT and QFT as a sum of multiplication of left-sided QFT can be derived.

Keywords : Convolution Theorem, Quaternion Fourier Transform, Quaternion Linear Canonical Transform

ABSTRAK

Transformasi Kanonikal Linier Quaternion (TKLQ) merupakan perluasan dari Transformasi Kanonikal Linier (TKL) pada aljabar quaternion yang berperan penting dalam bidang optik dan pemrosesan sinyal. TKLQ ini merupakan bentuk umum dari Transformasi Fourier Quaternion (TFQ). Berdasarkan hal tersebut, penelitian ini bertujuan untuk memformulasikan modifikasi teorema konvolusi untuk TKLQ sisi kiri berdasarkan hubungan antara TKLQ sisi kiri dan TFQ sisi kiri. Hasil penelitian menunjukkan dapat diturunkan formulasi modifikasi untuk teorema konvolusi TKLQ sisi kiri berdasarkan hubungan antara TKLQ sisi kiri dan TFQ sisi kiri yang dinyatakan sebagai penjumlahan dari perkalian TFQ untuk setiap fungsi.

Kata Kunci : Teorema Konvolusi, Transformasi Fourier Quaternion, Transformasi Kanonikal Linier Quaternion

I. PENDAHULUAN

Transformasi Kanonikal Linear (TKL) adalah transformasi integral linier dengan empat parameter $\{a, b, c, d\}$. TKL pertama kali diperkenalkan pada tahun 1970-an, dimana TKL merupakan bentuk generalisasi dari beberapa transformasi matematika seperti Transformasi Fourier (TF), Transformasi Fourier Fraksional (TFF) dan Transformasi Fresnel (Wei *et al.*, 2012).

Dalam penerapannya, TKL sangat berguna dalam analisis sistem optik dan pemrosesan sinyal khususnya sebagai alat yang sangat berguna di bidang optik, karena mampu menjelaskan efek dari setiap sistem fase kuadrat. Menurut Pei dkk. (2002) bahwa TKL juga berguna dalam menganalisa sistem radar, pemisahan sinyal (Sharma, 2006), desain filter (Barshan *et al.*, 1997) serta pengenalan pola. Kini beberapa studi telah dilakukan untuk pengembangan ilmu terkait TKL yang menghasilkan sifat-sifat penting dari TKL, yakni meliputi translasi, modulasi, konvolusi, dan korelasi serta prinsip ketidakpastian.

Selain itu, beberapa studi dan penelitian juga telah dilakukan terkait generalisasi TKL dalam bidang aljabar quaternion, yang disebut dengan Transformasi Kanonikal Linier Quaternion (TKLQ). Quaternion merupakan bilangan-bilangan kompleks yang diperluas untuk aljabar dimensi empat dan terdiri dari satu bagian real dan tiga bagian kompleks (Morais *et al.*, 2010). Adanya sifat non-komutatif terhadap perkalian bilangan quaternion menyebabkan TKLQ dibagi ke dalam tiga tipe, yakni TKLQ dua sisi, TKLQ sisi kanan dan TKLQ sisi kiri (Kou dan Morais, 2014).

Kini beberapa studi penting lainnya terkait TKLQ telah dilakukan seperti rumusan TKLQ sisi kanan dan sifat-sifatnya oleh Resnawati (2014) dan perilaku asimtotik TKLQ dirumuskan oleh Kou dan Morais (2014). Selanjutnya, aplikasi TKLQ untuk mempelajari filter pada frekuensi tinggi diperkenalkan oleh Bahri (2015) dan bukti sederhana untuk prinsip ketidakpastian transformasi kanonikal linear quaternion diformulasikan oleh Bahri dkk. (2016).

Kemudian Resnawati dan Musdalifah (2017) merumuskan hubungan antara TKLQ dengan TFQ yang menunjukkan bahwa TFQ merupakan kasus khusus dari TKLQ dan diformulasikan teorema konvolusi dari TKLQ sisi kanan sebagai hasil dari perkalian TFQ sisi kanan. TFQ itu sendiri merupakan perluasan dari Transformasi Fourier (TF) ke bidang aljabar qauternion. Sama halnya dengan TKLQ, maka TFQ ini juga dibedakan menjadi tiga jenis yakni TFQ dua sisi, TFQ sisi kanan dan TFQ sisi kiri (Pei, 2001). Dalam perkembangannya, telah dibuktikan beberapa sifat penting terkait TFQ seperti translasi, modulasi, differensiasi dan prinsip ketidakpastian (Bahri *et al.*, 2008). Terkait hal tersebut, penelitian ini bertujuan untuk memperoleh formulasi modifikasi teorema konvolusi TKLQ sisi kiri berdasarkan hubungan antara TKLQ sisi kiri dan TFQ sisi kiri.

II. METODE PENELITIAN

Penelitian ini bertempat di Jurusan Matematika FMIPA Universitas Tadulako. Rancangan penelitian ini berbentuk penelitian kualitatif dengan melakukan studi kepustakaan, serta mengumpulkan dan mengkaji materi-materi yang berkaitan dengan aljabar quaternion, TFQ dan TKLQ.

Penelitian ini dilakukan dengan melalui tahapan pertama yaitu mengkaji definisi TFQ sisi kiri dan TKLQ sisi kiri beserta sifat-sifatnya. Setelah itu, dilakukan pengkajian lagi terkait teorema konvolusi TKLQ sisi kiri dan sifat-sifatnya. Kemudian, merumuskan formulasi untuk modifikasi teorema konvolusi TKLQ sisi kiri berdasar atas hubungan antara TKLQ sisi kiri dan TFQ sisi kiri.

III. HASIL DAN PEMBAHASAN

Pada bagian ini akan diuraikan hasil modifikasi teorema konvolusi dari TKLQ sisi kiri, berdasarkan atas hubungan antara TFQ sisi kiri dan TKLQ sisi kiri. Oleh karena itu, bagian ini akan diawali dengan menyajikan definisi TFQ sisi kiri yaitu sebagai berikut.

Misalkan f suatu fungsi 2D bernilai quaternion. TFQ sisi kiri $\mathcal{F}_q : \mathbb{R}^2 \to \mathbb{H}$ dari $f \in L^2(\mathbb{R}^2;\mathbb{H})$, $\boldsymbol{x} = x_1\mathrm{e}_1 + x_2\mathrm{e}_2 \in \mathbb{R}^2$ dan $\boldsymbol{\omega} = \omega_1\mathrm{e}_1 + \omega_2\mathrm{e}_2 \in \mathbb{R}^2$ didefinisikan sebagai (Pei, 2001).

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu x_1\omega_1}\, e^{-\mu x_2\omega_2}\, f(\boldsymbol{x})\, d\boldsymbol{x}, \tag{1}$$

dimana $d\boldsymbol{x} = dx_1\, dx_2$ dan perkalian eksponensial quaternion $e^{-\mu x_1\omega_1}\, e^{-\mu x_2\omega_2}$ adalah kernel fourier quaternion.

3.1. Modifikasi TKLQ Sisi Kiri

Sebagai kasus khusus dari TKLQ sisi kiri yaitu jika $A_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ untuk $i = 1{,}2$ maka TKLQ sisi kiri dapat direduksi ke TFQ sisi kiri.

$$\begin{aligned}\mathcal{L}_A^{(l)}\{f\}(\omega) &= \int_{\mathbb{R}^2} K_{A_1}(x_1,\omega_1)\; K_{A_2}(x_2,\omega_2)\, f(\boldsymbol{x})\; d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mu\frac{1}{2}\left[\frac{a_1}{b_1}{x_1}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}{\omega_1}^2 - \frac{\pi}{2}\right]}\; \frac{1}{\sqrt{2\pi b_2}} e^{\mu\frac{1}{2}\left[\frac{a_2}{b_2}{x_2}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}{\omega_2}^2 - \frac{\pi}{2}\right]}\, f(\boldsymbol{x})\, d\boldsymbol{x}\end{aligned} \tag{2}$$

Substitusikan nilai $A_{1,2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ pada fungsi kernel di persamaan (2), sehingga diperoleh

$$\begin{aligned}\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{\mu\frac{1}{2}\left[-2x_1\omega_1 - \frac{\pi}{2}\right]}\; \frac{1}{\sqrt{2\pi}} e^{\mu\frac{1}{2}\left[-2x_2\omega_2 - \frac{\pi}{2}\right]}\; f(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{2\pi}} e^{-\mu x_1\omega_1}\;\; e^{-\mu x_2\omega_2} \frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{2\pi}}\; f(\boldsymbol{x})\; d\boldsymbol{x} \\ &= \frac{\mathbf{1}}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{-\mu\frac{\pi}{4}}\, e^{-\mu x_1\omega_1}\;\; e^{-\mu\frac{\pi}{4}}\, e^{-\mu x_2\omega_2}\, f(\boldsymbol{x})\;\; d\boldsymbol{x} \\ &= e^{-\mu\frac{\pi}{4}}\, e^{-\mu\frac{\pi}{4}}\, \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{\mathbb{R}^2} e^{-\mu x_1\omega_1}\;\; e^{-\mu x_2\omega_2}\, f(\boldsymbol{x})\;\; d\boldsymbol{x} \\ &= e^{-\mu\frac{\pi}{4}}\;\; e^{-\mu\frac{\pi}{4}}\;\; \mathcal{F}_q^{(l)}\{f\}(\boldsymbol{\omega}).\end{aligned}$$

Pada Teorema 1 diformulasikan hubungan antara TKLQ sisi kiri dan TFQ sisi kiri, dimana TKLQ sisi kiri dapat diekspresikan ke bentuk TFQ sisi kiri.

Teorema 1 : TKLQ sisi kiri dari fungsi quaternion $f \in L^2(\mathbb{R}^2;\mathbb{H})$ dengan matriks parameter $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ dan $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ dapat diekspresikan ke bentuk TFQ sisi kiri, sebagai berikut

$$\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega}) = e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{d_1}{b_1}\omega_1{}^2\right)}\, e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{d_2}{b_2}\omega_2{}^2\right)} \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right). \tag{3}$$

dimana :

$$\mathcal{F}_q^{(l)}\{g_f\}(\boldsymbol{\omega}) = \tilde{\mathcal{F}}(\boldsymbol{b\omega})$$

$$g_f(\boldsymbol{x}) = \frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_1}}\,\frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_2}}\,\tilde{f}(\boldsymbol{x})$$

$$\tilde{f}(\boldsymbol{x}) = e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_1}{b_1}x_1{}^2\right)}\, e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_2}{b_2}x_2{}^2\right)} f(\boldsymbol{x})$$

dengan

$$\tilde{\mathcal{F}}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} \frac{1}{\sqrt{(2\pi)^2}}\, e^{-\boldsymbol{\mu}x_1\frac{\omega_1}{b_1}}\, e^{-\boldsymbol{\mu}x_2\frac{\omega_2}{b_2}}\, g_f(\boldsymbol{x})\, d\boldsymbol{x}$$

$$\tilde{\mathcal{F}}(\boldsymbol{\omega}) = e^{-\boldsymbol{\mu}\frac{1}{2}\left(\frac{d_1}{b_1}\omega_1{}^2\right)}\, e^{-\boldsymbol{\mu}\frac{1}{2}\left(\frac{d_2}{b_2}\omega_2{}^2\right)}\, \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})$$

Bukti :

Dari definisi TKLQ sisi kiri pada persamaan (2), diperoleh

$$\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} K_{A_1}(x_1,\omega_1)\, K_{A_2}(x_2,\omega_2)\, f(\boldsymbol{x})\, d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]}\, \frac{1}{\sqrt{2\pi b_2}} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} f(\boldsymbol{x})\, d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\boldsymbol{\mu}\frac{x_1\omega_1}{b_1}} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{-\boldsymbol{\mu}\frac{\pi}{4}} \frac{1}{\sqrt{2\pi b_2}} e^{-\boldsymbol{\mu}\frac{x_2\omega_2}{b_2}} e^{\boldsymbol{\mu}\frac{1d_2}{2b_2}\omega_1{}^2}\, e^{-\boldsymbol{\mu}\frac{\pi}{4}} \left[e^{\boldsymbol{\mu}\frac{1a_1}{2b_1}x_1{}^2}\, e^{\boldsymbol{\mu}\frac{1a_2}{2b_2}x_2{}^2} f(\boldsymbol{x})\right] d\boldsymbol{x}$$

Misalkan $\tilde{f}(\boldsymbol{x}) = e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_1}{b_1}x_1{}^2\right)}\, e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_2}{b_2}x_2{}^2\right)} f(\boldsymbol{x})$, maka diperoleh

$$\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\boldsymbol{\mu}\frac{x_1\omega_1}{b_1}} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{-\boldsymbol{\mu}\frac{\pi}{4}}\, \frac{1}{\sqrt{2\pi b_2}} e^{-\boldsymbol{\mu}\frac{x_2\omega_2}{b_2}} e^{\boldsymbol{\mu}\frac{1d_2}{2b_2}\omega_2{}^2}\, e^{-\boldsymbol{\mu}\frac{\pi}{4}}\, \tilde{f}(\boldsymbol{x})\, d\boldsymbol{x} \tag{4}$$

Kalikan kedua ruas pada persamaan (4) dengan $e^{-\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{-\boldsymbol{\mu}\frac{1d_2}{2b_2}\omega_2{}^2}$, sehingga diperoleh :

$$e^{-\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{-\boldsymbol{\mu}\frac{1d_2}{2b_2}\omega_2{}^2}\, \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{-\boldsymbol{\mu}\frac{\pi}{4}}\, e^{-\boldsymbol{\mu}\frac{x_1\omega_1}{b_1}} \frac{1}{\sqrt{2\pi b_2}} e^{-\boldsymbol{\mu}\frac{x_2\omega_2}{b_2}}\, e^{-\boldsymbol{\mu}\frac{\pi}{4}}\, \tilde{f}(\boldsymbol{x})\, d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}}\, e^{-\boldsymbol{\mu}\frac{x_1\omega_1}{b_1}}\, \frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_1}}\, \frac{1}{\sqrt{2\pi}}\, e^{-\boldsymbol{\mu}\frac{x_2\omega_2}{b_2}}\, \frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_2}}\, \tilde{f}(\boldsymbol{x})\, d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}}\, e^{-\boldsymbol{\mu}\frac{x_1\omega_1}{b_1}}\, \frac{1}{\sqrt{2\pi}}\, e^{-\boldsymbol{\mu}\frac{x_2\omega_2}{b_2}} \left[\frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_1}}\, \frac{e^{-\boldsymbol{\mu}\frac{\pi}{4}}}{\sqrt{b_2}}\, \tilde{f}(\boldsymbol{x})\right] d\boldsymbol{x}. \tag{5}$$

Selanjutnya, dari persamaan (5) dimisalkan $g_f(\boldsymbol{x}) = \frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{b_1}} \frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{b_2}} \tilde{f}(\boldsymbol{x})$, maka diperoleh

$$e^{-\mu\frac{1d_1}{2b_1}\omega_1{}^2}\, e^{-\mu\frac{1d_1}{2b_1}\omega_1{}^2}\, \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} \; e^{-\mu\frac{x_1\omega_1}{b_1}} \frac{1}{\sqrt{2\pi}} \; e^{-\mu\frac{x_2\omega_2}{b_2}} \; g_f(\boldsymbol{x})\, d\boldsymbol{x}$$

$$= \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-\mu\frac{\omega_1}{b_1}x_1} \; e^{-\mu\frac{\omega_2}{b_2}x_2} \; g_f(\boldsymbol{x}) \; d\boldsymbol{x}$$

$$= \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right),$$

atau dapat dituliskan sebagai

$$\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega}) = e^{\mu\frac{1d_1}{2b_1}\omega_1{}^2}\, e^{\mu\frac{1d_1}{2b_1}\omega_1{}^2} \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right).$$

Dengan demikian teorema 1 terbukti.

Selain itu berdasarkan hubungan TKLQ dan TFQ sisi kiri pula, dapat diturunkan formulasi untuk invers dari TKLQ sisi kiri, yaitu sebagai berikut.

Teorema 2 : Jika $f \in L^2(\mathbb{R}^2;\mathbb{H})$ dan $\mathcal{L}_A^{(l)}\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$, maka invers TKLQ sisi kiri dapat diturunkan dari TFQ.

Bukti :

$$\begin{aligned} g_f(\boldsymbol{x}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\omega_1}\; e^{\mu x_2\omega_2}\; \mathcal{F}_q^{(l)}\{g_f\}(\boldsymbol{\omega})\, d\boldsymbol{\omega} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\omega_1}\; e^{\mu x_2\omega_2}\; \tilde{\mathcal{F}}(\boldsymbol{b\omega})\, d\boldsymbol{\omega} \\ &= \frac{1}{b_1b_2}\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\left(\frac{\omega_1}{b_1}\right)}\; e^{\mu x_2\left(\frac{\omega_2}{b_2}\right)} \tilde{\mathcal{F}}(\boldsymbol{\omega})\; d\boldsymbol{\omega} \\ &= \frac{1}{b_1b_2 2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\left(\frac{\omega_1}{b_1}\right)}\; e^{\mu x_2\left(\frac{\omega_2}{b_2}\right)}\; e^{-\mu\frac{1d_1}{2b_1}\omega_1{}^2}\; e^{-\mu\frac{1d_2}{2b_2}\omega_2{}^2}\; \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})d\boldsymbol{\omega} \end{aligned}$$

Persamaan di atas dapat dituliskan sebagai :

$$\frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{b_1}} \frac{e^{-\mu\frac{\pi}{4}}}{\sqrt{b_2}} \; e^{\mu\frac{1}{2}\left(\frac{a_1}{b_1}x_1{}^2\right)}\; e^{\mu\frac{1}{2}\left(\frac{a_2}{b_2}x_2{}^2\right)} f(\boldsymbol{x})$$

$$= \frac{1}{b_1b_2 2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\left(\frac{\omega_1}{b_1}\right)}\; e^{\mu x_2\left(\frac{\omega_2}{b_2}\right)}\; e^{-\mu\frac{1d_1}{2b_1}\omega_1{}^2}\; e^{-\mu\frac{1d_2}{2b_2}\omega_2{}^2}\; \mathcal{L}_A^{(l)}\{f\}(\omega)\, d\boldsymbol{\omega}$$

sehingga

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{\sqrt{b_1}\,\sqrt{b_2}}{b_1b_2 2\pi}\int_{\mathbb{R}^2} e^{\mu x_1\left(\frac{\omega_1}{b_1}\right)}\; e^{\mu x_2\left(\frac{\omega_2}{b_2}\right)}\; e^{-\mu\frac{1d_1}{2b_1}\omega_1{}^2}\; e^{-\mu\frac{1d_2}{2b_2}\omega_2{}^2} \\ &\quad \cdot e^{-\mu\frac{1}{2}\left(\frac{a_1}{b_1}x_1{}^2\right)}\; e^{-\mu\frac{1}{2}\left(\frac{a_2}{b_2}x_2{}^2\right)}\; e^{\mu\frac{\pi}{4}}\, e^{\mu\frac{\pi}{4}}\; \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})\, d\boldsymbol{\omega} \\ &= \frac{\sqrt{b_1}\,\sqrt{b_2}}{b_1b_2 2\pi}\int_{\mathbb{R}^2} e^{-\mu\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2-\frac{2}{b_1}x_1\omega_1+\frac{d_1}{b_2}\omega_1{}^2-\frac{\pi}{2}\right]}\; e^{-\mu\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}\right]}\; \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\mu\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2-\frac{2}{b_1}x_1\omega_1+\frac{d_1}{b_2}\omega_1{}^2-\frac{\pi}{2}\right]} \end{aligned}$$

$$\cdot \frac{1}{\sqrt{2\pi b_2}} e^{-\mu\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}\right]} \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})d\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2} K_{A_1^{-1}}(x_1,\omega_1)\ K_{A_2^{-1}}(x_2,\omega_2)\ \mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})\ d\boldsymbol{\omega}$$

3.2. Modifikasi Teorema TKLQ Sisi Kiri dari Konvolusi Fungsi Quaternion

Pada bagian ini teorema TKLQ sisi kiri dari konvolusi fungsi quaternion akan dimodifikasi berdasarkan hubungan TKLQ sisi kiri dan TFQ sisi kiri.

Teorema 4 : Misalkan $f,g \in L^2(\mathbb{R}^2;\mathbb{H})$ adalah dua fungsi yang bernilai quaternion. Maka modifikasi TKLQ sisi kiri dari konvolusi $f,g \in L^2(\mathbb{R}^2;\mathbb{H})$ dapat dinyatakan sebagai berikut

$$\begin{aligned}\mathcal{L}_A^{(l)}\{f*g\}(\boldsymbol{\omega}) &= e^{\mu\frac{1d_1}{2b_1}\omega_1{}^2} e^{\mu\frac{1d_1}{2b_1}\omega_1{}^2}\Big[\mathcal{F}_q^{(l)}\{\tilde{g}_0\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\\ &+\mathcal{F}_q^{(l)}\{\tilde{g}_1\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i}+\mathcal{F}_q^{(l)}\{\tilde{g}_2\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{j}\\ &+\mathcal{F}_q^{(l)}\{\tilde{g}_3\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{k}\Big]. \qquad (6)\end{aligned}$$

Bukti :

$$\begin{aligned}\mathcal{L}_A^{(l)}\{f*g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}} e^{\mu\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2-\frac{2}{b_1}x_1\omega_1+\frac{d_1}{b_1}\omega_1{}^2-\frac{\pi}{2}\right]} e^{\frac{a_1}{b_1}\mu t_1(t_1-x_1)}\\ &\cdot\frac{1}{\sqrt{2\pi b_2}} e^{\mu\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}\right]} e^{\frac{a_2}{b_2}\mu t_2(t_2-x_2)} f(\boldsymbol{t})g(\boldsymbol{x}-\boldsymbol{t})\,d\boldsymbol{x}\,d\boldsymbol{t} \qquad (7)\end{aligned}$$

Misalkan $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{t}$, maka persamaan (7) dapat dituliskan menjadi

$$\begin{aligned}&\mathcal{L}_A^{(l)}\{f*g\}(\boldsymbol{\omega})\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}} e^{\mu\frac{1}{2}\left[\frac{a_1}{b_1}(y_1+t_1)^2-\frac{2}{b_1}(y_1+t_1)\omega_1+\frac{d_1}{b_1}\omega_1{}^2-\frac{\pi}{2}\right]} e^{\frac{a_1}{b_1}(-\mu t_1y_1)}\\ &\cdot\frac{1}{\sqrt{2\pi b_2}} e^{\mu\frac{1}{2}\left[\frac{a_2}{b_2}(y_2+t_2)^2-\frac{2}{b_2}(y_2+t_2)\omega_2+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}\right]} e^{\frac{a_1}{b_1}(-\mu t_2y_2)} f(\boldsymbol{t})g(\boldsymbol{y})\,d\boldsymbol{t}\,d\boldsymbol{y}\\ &=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}} e^{\mu\frac{1}{2}\left(\frac{a_1}{b_1}(y_1{}^2+2y_1t_1+t_1{}^2)\right)-\frac{2}{b_1}(y_1\omega_1+t_1\omega_1)+\frac{d_1}{b_1}\omega_1{}^2-\frac{\pi}{2}} e^{\frac{a_1}{b_1}(-\mu t_1y_1)}\\ &\cdot\frac{1}{\sqrt{2\pi b_2}} e^{\mu\frac{1}{2}\left(\frac{a_2}{b_2}(y_2{}^2+2y_2t_2+t_2{}^2)\right)-\frac{2}{b_2}(y_2\omega_2+t_2\omega_2)+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}} e^{\frac{a_1}{b_1}(-\mu t_2y_2)} f(\boldsymbol{t})g(\boldsymbol{y})\,d\boldsymbol{t}\,d\boldsymbol{y}\\ &=\int_{\mathbb{R}^2} e^{\mu\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\mu\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\mu\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\mu\frac{1}{2}\frac{a_2}{b_2}(y_2^2)}\\ &\cdot\Bigg[\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}} e^{\mu\frac{1}{2}\left[\frac{a_1}{b_1}t_1{}^2-\frac{2}{b_1}(t_1\omega_1)+\frac{d_1}{b_1}\omega_1{}^2-\frac{\pi}{2}\right]}\\ &\cdot\frac{1}{\sqrt{2\pi b_2}} e^{\mu\frac{1}{2}\left[\frac{a_2}{b_2}t_2{}^2-\frac{2}{b_2}(t_2\omega_2)+\frac{d_2}{b_2}\omega_2{}^2-\frac{\pi}{2}\right]} f(\boldsymbol{t})\,d\boldsymbol{t}\Bigg]\, g(\boldsymbol{y})\,d\boldsymbol{y}\\ &=\int_{\mathbb{R}^2} e^{\mu\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\mu\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\mu\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\mu\frac{1}{2}\frac{a_2}{b_2}(y_2^2)}\mathcal{L}_A^{(l)}\{f\}(\boldsymbol{\omega})\,g(\boldsymbol{y})\,d\boldsymbol{y}.\end{aligned}$$

Selanjutnya, berdasarkan Teorema 1 maka persamaan diatas dapat dituliskan sebagai

$$\mathcal{L}_A^{(l)}\{f * g\}(\boldsymbol{\omega})$$
$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_2}{b_2}(y_2^2)}$$
$$\cdot e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) g(\boldsymbol{y})\, d\boldsymbol{y}.$$
$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_2}{b_2}(y_2^2)} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2}$$
$$\cdot \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) [g(\boldsymbol{y}) + \boldsymbol{i}g_1(\boldsymbol{y}) + \boldsymbol{j}g_2(\boldsymbol{y}) + \boldsymbol{k}g_3(\boldsymbol{y})]\, d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_2}{b_2}(y_2^2)} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2}$$
$$\cdot [\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) g_0(\boldsymbol{y}) + \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{i}g_1(\boldsymbol{y}) + \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j}g_2(\boldsymbol{y}).$$
$$+\mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{k}g_3(\boldsymbol{y})]\, d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_1}{b_1}(y_1{}^2)} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\boldsymbol{\mu}\frac{1}{2}\frac{a_2}{b_2}(y_2^2)} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2}$$
$$\left[g_0(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) + g_1(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i}\right.$$
$$\left.+g_2(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} + \boldsymbol{k}g_3(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right]\, d\boldsymbol{y}. \tag{8}$$

Dimisalkan $\tilde{g}_n(\boldsymbol{y}) = e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_1}{b_1}y_1{}^2\right)} e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_2}{b_2}y_2{}^2\right)} g(\boldsymbol{y})$ untuk $n = 0,1,2,3$. Maka persamaan (8) dapat dituliskan dengan

$$\mathcal{L}_A^{(l)}\{f * g\}(\boldsymbol{\omega})$$
$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_1}\right)(y_1\omega_1)} e^{\boldsymbol{\mu}\frac{1}{2}\left(-\frac{2}{b_2}\right)(y_2\omega_2)} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} \left[\tilde{g}_0(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right.$$
$$\left[+\tilde{g}_1(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i} + \tilde{g}_2(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} + \tilde{g}_3(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{k}\right]\, d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{\mu}y_1\left(\frac{\omega_1}{b_1}\right)} e^{-\boldsymbol{\mu}y_2\left(\frac{\omega_2}{b_2}\right)} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} \left[\tilde{g}_0(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) + \tilde{g}_1(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i}\right.$$
$$\left.+\tilde{g}_2(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} + \tilde{g}_3(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{k}\right]\, d\boldsymbol{y}. \tag{9}$$

Dengan menerapkan definisi TFQ sisi kiri pada persamaan (1), maka persamaan (9) dapat dituliskan sebagai

$$\mathcal{L}_A^{(l)}\{f * g\}(\boldsymbol{\omega})$$
$$= e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} \left[\mathcal{F}_q^{(l)}\{\tilde{g}_0\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) + \mathcal{F}_q^{(l)}\{\tilde{g}_1\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i}\right.$$
$$\left.+\mathcal{F}_q^{(l)}\{\tilde{g}_2\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} + \mathcal{F}_q^{(l)}\{\tilde{g}_3\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_f\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{k}\right].$$

Dengan demikian persamaan (6) terbukti, yang menunjukkan bahwa hasil modifikasi TKLQ sisi kiri dari konvolusi fungsi quaternion merupakan penjumlahan dari hasil kali TFQ sisi kiri untuk masing-masing fungsi.

3.3. Modifikasi Teorema TKLQ Sisi Kiri dari Konvolusi Fungsi Translasi

Berikut akan diformulasikan modfikasi teorema TKLQ sisi kiri dari konvolusi fungsi translasi.

Teorema 4 : Misalkan $f, g \in L^2(\mathbb{R}^2\,;\,\mathbb{H})$ adalah dua fungsi bernilai quaternion, maka modifikasi TKLQ sisi kiri dari konjugat konvolusi adalah

$$\begin{aligned}\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega}) \quad &= e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1d_1}{2b_1}\omega_1{}^2} \left\{\mathcal{F}_q^{(l)}\{\tilde{f}_0\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right. \\ &\quad -\mathcal{F}_q^{(l)}\{\tilde{f}_1\}\,\mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{i} - \mathcal{F}_q^{(l)}\{\tilde{f}_2\}\,\mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{j} \\ &\quad \left. -\mathcal{F}_q^{(l)}\{\tilde{f}_3\}\,\mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\boldsymbol{k}\right\}. \qquad (10)\end{aligned}$$

Bukti :

$$\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} \left(\overline{f * g}\right)(\boldsymbol{x})\; d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} \left(\overline{g} * \overline{f}\right)(\boldsymbol{x})\; d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]}$$

$$\cdot \left[\int_{\mathbb{R}^2} e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(t_1 - x_1)}\; e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(t_2 - x_2)}\; \overline{g}(\boldsymbol{t})\,\overline{f}(\boldsymbol{x} - \boldsymbol{t})\; d\boldsymbol{t}\right] d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}x_1{}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}x_2{}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(t_1 - x_1)}$$

$$\cdot\; e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(t_2 - x_2)}\; \overline{g}(\boldsymbol{t})\,\overline{f}(\boldsymbol{x} - \boldsymbol{t})\; d\boldsymbol{x}\, d\boldsymbol{t}.$$

Misalkan $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{t}$, maka diperoleh

$$\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1 + t_1)^2 - \frac{2}{b_1}(y_1 + t_1)\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2 + t_2)^2 - \frac{2}{b_2}(y_2 + t_2)\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]}$$

$$\cdot\, e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)}\; e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)}\; \overline{g}(\boldsymbol{t})\,\overline{f}(\boldsymbol{y})\; d\boldsymbol{y}\, d\boldsymbol{t}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2 + 2y_1t_1 + t_1{}^2) - \frac{2}{b_1}(y_1\omega_1 + t_1\omega_1) + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]}$$

$$\cdot \frac{1}{\sqrt{2\pi b_2}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2 + 2y_2t_2 + t_2{}^2) - \frac{2}{b_2}(y_2\omega_2 + t_2\omega_2) + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)} e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)}\; \overline{g}(\boldsymbol{t})\,\overline{f}(\boldsymbol{y})\; d\boldsymbol{y}\, d\boldsymbol{t}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}t_1{}^2 - \frac{2}{b_1}t_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \frac{1}{\sqrt{2\pi b_1}}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}t_2{}^2 - \frac{2}{b_2}t_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]}$$

$$\cdot\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2 + 2y_1t_1) - \frac{2}{b_1}(y_1\omega_1)\right]}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2 + 2y_2t_2) - \frac{2}{b_2}(y_2\omega_2)\right]}\, e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)}\; e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)} \overline{g}(\boldsymbol{t})\,\overline{f}(\boldsymbol{y})\; d\boldsymbol{y}\, d\boldsymbol{t}$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2 + 2y_1t_1) - \frac{2}{b_1}(y_1\omega_1)\right]}\, e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2 + 2y_2t_2) - \frac{2}{b_2}(y_2\omega_2)\right]}\, e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)}$$

$$\cdot e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1} t_1{}^2 - \frac{2}{b_1} t_1\omega_1 + \frac{d_1}{b_1}\omega_1{}^2 - \frac{\pi}{2}\right]} \right.$$

$$\left. \cdot \frac{1}{\sqrt{2\pi b_1}} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1} t_2{}^2 - \frac{2}{b_2} t_2\omega_2 + \frac{d_2}{b_2}\omega_2{}^2 - \frac{\pi}{2}\right]} \overline{g}(\boldsymbol{t})\ d\boldsymbol{t} \right] \cdot \overline{f}(\boldsymbol{y})\, d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2+2y_1t_1) - \frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2+2y_2t_2) - \frac{2}{b_2}(y_2\omega_2)\right]}$$

$$\cdot e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)} e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)} \mathcal{L}_A^{(l)}\{\overline{g}\}(\boldsymbol{\omega})\, \overline{f}(\boldsymbol{y})\, d\boldsymbol{y}. \tag{11}$$

Dengan menerapkan Teorema 1 sebagai hasil modifikasi TKLQ sisi kiri pada persamaan (11), maka diperoleh

$$\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2+2y_1t_1) - \frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2+2y_2t_2) - \frac{2}{b_2}(y_2\omega_2)\right]} e^{\frac{a_1}{b_1}\boldsymbol{\mu} t_1(-y_1)} e^{\frac{a_2}{b_2}\boldsymbol{\mu} t_2(-y_2)}$$

$$\cdot e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \overline{f}(\boldsymbol{y})\, d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2) - \frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2) - \frac{2}{b_2}(y_2\omega_2)\right]} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \overline{f}(\boldsymbol{y})\, d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2) - \frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2) - \frac{2}{b_2}(y_2\omega_2)\right]} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \left\{\mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) f_0(\boldsymbol{y})\right.$$

$$\left. -\mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{i} f_1(\boldsymbol{y}) - \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} f_2(\boldsymbol{y}) - \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{k}\, f_3(\boldsymbol{y})\right\} d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_1}{b_1}(y_1{}^2) - \frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[\frac{a_2}{b_2}(y_2{}^2) - \frac{2}{b_2}(y_2\omega_2)\right]} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \left\{f_0(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right.$$

$$\left. -f_1(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{i} - f_2(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} - f_3(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{k}\right\} d\boldsymbol{y} \tag{12}$$

Pada persamaan (12) dimisalkan $\tilde{f}_n(\boldsymbol{y}) = e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_1}{b_1}y_1{}^2\right)} e^{\boldsymbol{\mu}\frac{1}{2}\left(\frac{a_2}{b_2}y_2{}^2\right)} f(\boldsymbol{y})$ untuk $n = 0,1,2,3$. Maka persamaan (12) diubah menjadi

$$\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2} e^{\boldsymbol{\mu}\frac{1}{2}\left[-\frac{2}{b_1}(y_1\omega_1)\right]} e^{\boldsymbol{\mu}\frac{1}{2}\left[-\frac{2}{b_2}(y_2\omega_2)\right]} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \left\{\tilde{f}_0(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right.$$

$$\left. -\tilde{f}_1(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{i} - \tilde{f}_2(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j} - \tilde{f}_3(\boldsymbol{y})\, \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{k}\right\} d\boldsymbol{y} \tag{13}$$

Dengan menerapkan Teorema 1 sebagai hasil modifikasi TKLQ sisi kiri pada persamaan (13), maka diperoleh

$$\mathcal{L}_A^{(l)}\{\overline{f * g}\}(\boldsymbol{\omega}) = e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} e^{\boldsymbol{\mu}\frac{1 d_1}{2 b_1}\omega_1{}^2} \left\{\mathcal{F}_q^{(l)}\{\tilde{f}_0\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right)\right.$$
$$-\mathcal{F}_q^{(l)}\{\tilde{f}_1\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{i} - \mathcal{F}_q^{(l)}\{\tilde{f}_2\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{j}$$
$$\left. -\mathcal{F}_q^{(l)}\{\tilde{f}_3\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \mathcal{F}_q^{(l)}\{g_g\}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{b}}\right) \boldsymbol{k}\right\}.$$

Dengan demikian teorema modifikasi TKLQ sisi kiri dari konjugat konvolusi pada persamaan (10) terbukti.

IV. KESIMPULAN

Berdasarkan hasil dan pembahasan, diperoleh rumusan untuk modifikasi teorema konvolusi TKLQ sisi kiri berdasarkan hubungan antara TKLQ sisi kiri dan TFQ sisi kiri.

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